1 Over-Lapping Generations

- What the heck is OLG?
  - Infinite succession of agents who live for two periods
  - Each period there $N_{t-1}$ old agents and $N_t$ young agents
  - At the start of the next period:
    * the $N_t$ young agents will become old
    * the $N_{t-1}$ old agents will be replaced with $N_{t+1}$ young agents
  - Agents only care about consumption while alive
  - Markets are incomplete: no trading at time 0.

- Why learn another model?
  - Presence of retired people (ie too old to work) implies a strong motive for saving
  - Competitive equilibrium is not always Pareto-Optimal, due to presence of an initial old generation.
    * in absence of time-0 trading, initial old agents must consume their endowment, therefore young people must also consume theirs.
  - Potential role for money or government transfers to restore efficiency
  - Model can be extended (altruism, stochastic lifetimes) to build useful theories of family dynamics (Barro-Becker), deficit finance(Barro, 1974)...

- Overview of presentation:
  - start by treating OLG as an example of complete markets; allow time-0 trading (unusual)
    * derive stationary equilibria: constant interest rate
    * non-stationary equilibria:
derive the savings function, represent in terms of the “offer curve”

- graphical analysis of Lucas-tree economy, government spending
  - shut down time-0 markets to analyse the traditional OLG problem
  - applications to public pensions (= social security)

2 Environment and Notation

- Agents \( i = \{0, 1, 2, 3\ldots\} \) born at time \( t = i \)
- There are \( N_i \) agents of each vintage
- Endowment \( y^i_t \) in period \( t \in \{0, 1, 2, 3\ldots\} \)
- Agents consume at periods \( i, i+1 \), bundle denoted \( c^i_i, c^i_{i+1} \)
- Preferences
  \[
  U(c^i_i, c^i_{i+1}) = u(c^i_i) + \beta u(c^i_{i+1})
  \]
- Specialise to follow LS:
  - equal numbers: \( N_i = N_{i+1} = \ldots \)
  - no discounting: \( \beta = 1 \)
  - endowments of agent \( i \) are only non-zero in periods \( i, i + 1 \)

3 OLG with Time-0 Trading

- market at time 0 with trading in goods for all periods; price vector \( \{q^0_0, q^0_1 \ldots q^0_t \ldots\} \)
- Household problem:
  \[
  \max_{c^i_i, c^i_{i+1}} \{U(c^i_i, c^i_{i+1})\}
  \]
  subject to budget constraint :
  \[
  \sum_{t=0} q^0_t c^i_t \leq \sum_{t=0} q^0_t y^i_t \quad (1)
  \]
- FOC:
  \[
  u'(c^i_i) = u'(c^i_{i+1}) \frac{q^0_{t+1}}{q^0_t} \quad (2)
  \]
- FC:
  \[
  c^i_i + c^{i-1}_i = y^i_i + y^{i-1}_i \quad (3)
  \]
  Note that in a stationary allocation consumption for each group is constant over time:
  \[
  c^i_i = c^i_{i+1} = c_o
  \]
3.1 Example 9.2.1

- Note that the endowment is smaller when old, so if the interest rate is sufficiently high, people will save for the future.
  - Derive condition on \( q \) for saving to occur.
  - However the old will not save, regardless, so by (3) there must be some good wasted. Can this be efficient?
  - The value of the aggregate endowment is infinite; this violates the assumptions of the First welfare theorem
  - How can we rule out this equilibrium? If the endowment is positive only for a finite number of periods
  - In that case, autarky is the only equilibrium and is efficient.

4 The Offer Curve

- For any given \( \alpha = \frac{q_i}{q_{i+1}} \), there is a unique optimal bundle \((c_i, c_{i+1})\) that the household would choose
- The offer curve gives all the bundles \((c_i, c_{i+1})\) for which there exists some \( \alpha \) that would make the bundle optimal
- Diagram 9.2.1

4.1 Non-Stationary equilibria

- In general, with \( \alpha \) varying over time the offer curve will be a first-order difference equation in \( \alpha \)
- In Diagram 9.2.2 we can trace out the time path of \( \alpha \)

4.1.1 Log Example (9.2.5)

- Suppose \( u(c) = \ln(c) \)
- FOC+BC implies equal expenditure shares on goods:
  \[
  q_i c_i^i = q_{i+1} c_{i+1}^i = \frac{1 - \epsilon + \alpha_i}{2}
  \]
  \[
  c_{i+1}^i = \frac{c_i^i}{\alpha_i}
  \]
- FC:
  \[
  \frac{1 - \epsilon + \alpha_i}{2} + \frac{1 - \epsilon + \alpha_{i-1}}{2\alpha_{i-1}} = 1
  \]
- Rearange:
  \[
  \alpha_i = \epsilon^{-1} - \frac{\epsilon^{-1} - 1}{\alpha_{i-1}}
  \]
5 Sequential Trading

Here we part ways with LS. The goal is to discuss public pensions in the context of a production economy. First we consider the welfare implications of ruling out time-0 trading, then we introduce a capital stock, which allows inter-temporal reallocation, and finally consider the conditions under which public pensions might benefit old people. The analysis is driven by the impact of public pensions on savings and hence on the capital stock, which is why we need to extend the model.

5.1 An OLG exchange economy

Time is discrete and continues forever from $t = 0, 1, 2, \ldots$. The economy consists of many identical individuals who live for two periods: in the first period of their life they are young and in the second period they are old. At every date $t$ a new generation $G_t$ of agents is born. Each generation has the same mass; there is no population growth.

There is also the initial generation $G_0$ of agents who are old at $t = 0$ and hence only live for one period. The life-time utility function $u(c_t^1, c_t^2)$ of an agent born at time $t$ is strictly increasing and quasi-concave in consumption when young and old, respectively. Agents in each generation $G_t, t > 0$ have endowments $(e_1, e_2)$, while the initial old agent has endowment $e_2$.

We can represent the MRS as

$$\mu \left( c_t^1, c_{t+1}^1 \right) \equiv \frac{u_1 \left( c_t^1, c_{t+1}^1 \right)}{u_2 \left( c_t^1, c_{t+1}^1 \right)}$$

5.1.1 Pareto-Optimal Allocations

The resource constraint of this economy is

$$c_t^1 + c_{t-1}^1 = e_1 + e_2$$

A PO allocation is a feasible consumption sequence $c = \{c_t^1, c_{t+1}^1\}_{t=0}^\infty$ such that there exists no other feasible allocation $\hat{c}$ where:

1. $\hat{c}_{02} \geq c_1^0$
2. $u(\hat{c}_{11}, \hat{c}_{12}) \geq u(c_t^1, c_{t+1}^1)$ for all $t > 0$
3. At least one of the above inequalities is strict for some $t$.

Consider the planner’s problem for $t > 0$:

$$\max_{c_t^1, c_{t+1}^1} u(c_t^1, c_{t+1}^1)$$

subject to the resource constraint (FC). The FOC for this implies that at an interior solution, $\mu = 1$:

$$u_1 \left( c_t^1, c_{t+1}^1 \right) - u_2 \left( c_t^1, c_{t+1}^1 \right) = 0$$
Suppose that at the endowment, $\mu < 1$. In other words $u_1(e_1, e_2) < u_2(e_1, e_2)$, so DMU implies the planner would like to transfer consumption from young to old. This makes everyone better off; the initial old get more consumption, and everyone gets an optimal allocation. So the planner can do better than the endowment. To do this requires only that for any generation of old, there be a generation of young to pay for the transfer.

Now suppose instead that $\mu > 1$ at the endowment. This means the planner can make all generations after $t = 0$ better off by reducing their consumption when old and increasing it when young. However this is NOT a Pareto improvement! The FC in the first period implies that the initial old generation gets less consumption, which violates condition 1 of our definition. Since the planner can’t beat the initial endowment, that endowment point is Pareto Optimal.

### 5.1.2 Competitive Equilibrium

We are going to define a recursive competitive equilibrium for this economy; instead of defining it in terms of the infinite allocation sequence, we can reduce it to two periods, by assuming that agents can borrow and lend freely. Given a price sequence $\{q_t\}_{t=0}^{\infty}$ the household budget constraint can be expressed in terms of the ratio $R_t = q_t/q_{t+1}$:

\[
\begin{align*}
q_t c_t^t + q_{t+1} c_{t+1}^t &= q_t e_1 + q_{t+1} e_2 \\
\frac{q_t}{q_{t+1}} c_t^t + c_{t+1}^t &= q_t e_1 + e_2 \\
R_t c_t^t + c_{t+1}^t &= R_t e_1 + e_2
\end{align*}
\]

A **Recursive Competitive Equilibrium** of this economy is a sequence $\{R_t, c_t^t, c_{t+1}^t\}$ such that

1. (Optimality condition):
   
   (a) the consumption pair $c_t^t, c_{t+1}^t \geq 0$ maximizes $u(c_t^t, c_{t+1}^t)$ for generation $G_t$, given $R_t$ subject to the household budget constraint.
   
   (b) for $t = 0$, $c_0^0 = e_2$; the initial old in $G_0$ consume their endowments

2. Markets clear:

\[c_t^t + c_{t+1}^t = e_1 + e_2\]

The only equilibrium allocation is autarky: $(c_t^t, c_{t+1}^t) = (e_1, e_2)$. Since everyone is the same, there is no trade within a generation, and since $c_0^0 = e_2$ the feasibility constraint implies $c_1^0 = e_1$. The budget constraint then implies $c_{12} = e_2$. By the same reasoning, therefore we get that $c_{t+1}^t = e_2$ and $c_{21} = e_1$. The equilibrium interest rate is therefore

\[R_t^* = \mu (e_1, e_2) = \frac{u_1 (e_1, e_2)}{u_2 (e_1, e_2)}\]
5.1.3 Is the RCE Pareto optimal?

Assume: \((e_1, e_2) = (1, 0)\), \(U(c_t^t, c_t^{t+1}) = c_t^t + c_t^{t+1}\). It is possible to make a Pareto improvement here by giving all the endowment to the old people. This allocation \((c_t^t, c_t^{t+1}) = (0, 1)\) leaves young people indifferent and the initial old much better off. The welfare theorem therefore does not apply because the equilibrium allocation is not efficient.

The problem is that there is an infinite number of agents so we can have allocations in which some people appear to be spending more than their income. The bill just gets passed to the next generation. If there was a last generation, this would break down as they would be worse off!

Because the budget line in this case has slope -1, it is easy to show that the equilibrium is Pareto optimal iff the MRS \(\mu(e_1, e_2) > 1\). (ie at the endowment point).

For instance, consider the case \(e_1 = 1\). If this condition is violated, so that \(\mu(e_1, e_2) < 1\), then the indifference curve cuts the budget line at the endowment, and so a better allocation is feasible (eg the one that makes the IC tangent to the BC). This allocation involves giving consumption to the initial old, so they are better off too.

On the other hand if \(\mu(e_1, e_2) > 1\) then getting to the higher indifference curve requires taking away consumption from the old and giving it to the young. This would make the initial old worse off. Therefore in this case, the no-trade equilibrium is Pareto optimal.

With population growth at rate \(\gamma > 0\) we have \(n_t = \gamma n_{t-1}\). The feasibility constraint implies society can transfer resources across generations at rate \(\gamma\):

\[
\gamma c^t_t + c^{t-1}_t = \gamma e_1 + e_2
\]

so equilibrium is efficient iff \(\mu(e_1, e_2) > \gamma\).

5.2 Production in the OLG Model

5.2.1 Environment

Let the number of people in generation \(t\) be \(N_t\). The population grows at rate \(n\) so \(N_t = N_0 (1 + n)^t\). There is an initial capital stock \(K_0\). The technology is given by \(Y_t = F(K_t, N_t)\), where \(Y_t\) is output and \(K_t\) and \(N_t\) are the capital and labor inputs, respectively. Assume that the production function \(F\) is strictly increasing, strictly quasi-concave, twice differentiable, and homogeneous of degree one (ie satisfies CRS and DMR).

5.2.2 Markets

Young agents sell their labor to firms and receive labor income \(w_t\). They save in the form of capital accumulation. Old agents rent capital to firms and then convert the capital into consumption goods which they consume. Let \(r_{t+1}\) be the net interest rate: \(r_{t+1} = q_t / q_{t+1} - 1\). The representative firm maximizes
profits by producing consumption goods, and renting capital and hiring labor as inputs.

5.2.3 Preferences
Let lifetime utility be separable:

\[ U(c_t, c_{t+1}) = u(c_t) + \frac{1}{1+\theta} u(c_{t+1}) \]

where \( \theta > 0 \) is the rate of time preference and \( u \) is concave.

5.2.4 Pareto-optimal Allocations
The resource constraint is

\[ F(K_t, N_t) + K_t = c_t N_t + c_{t-1} N_{t-1} + K_{t+1} \]

Denote per-capita terms by lower case; the FC can be expressed as:

\[ \frac{F(K_t, N_t)}{N_t} + \frac{K_t}{N_t} = \frac{c_t N_t}{N_t} + \frac{c_{t-1} N_{t-1}}{N_t} + \frac{K_{t+1}}{N_t} \]

which simplifies (using CRS) to:

\[ f(k_t) + k_t = c_1 + \frac{c_2}{1+n} + (1+n) k_{t+1} \]

To keep things simple, suppose that there is a steady state, so that all per capita quantities are constant. The FC becomes

\[ f(k) + k = c_1 + \frac{c_2}{1+n} + (1+n) k \]
\[ f(k) - nk = c_1 + \frac{c_2}{1+n} \]

In the steady state, for all but the current old, the planner chooses \((c_1, c_2)\), and \( k \) to solve

\[ \max_{c_1, c_2} U(c_1, c_2) \]

subject to the FC. The FOC for an interior optimum are

\[ \mu(c_1, c_2) = \frac{U_1}{U_2} = \frac{1 + \theta}{u'(c_1)} \frac{u'(c_2)}{u'(c_2)} = 1 + n \]
\[ f'(k) = n \]

We shall refer to the capital stock level that solves this problem as the “Golden Rule” level of the capital stock, \( k_{GR} \). Note that it does NOT take into account the utility of the initial old. To be pareto-optimal, an allocation with savings must find a way to deliver the savings of the initial young to the initial old. Therefore the savings rate that leads to a steady-state capital stock level equal to \( k_{GR} \) will be known as the “Golden Rule savings rate”.

7
5.2.5 The RCE

We need to see now under what conditions the RCE gives a PO allocation.

**The Household Problem** Individuals receive labor income $w_t$ in the first part of life and save $s_t$ for 2nd-period consumption. The budget constraints are

$$
c_t = w_t - s_t
$$

$$
c_{t+1} = (1 + r_{t+1}) s_t
$$

The Euler equation is

$$
\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1 + r_{t+1}}{1 + \theta}
$$

Suppose that

$$
u'(c_t) = c^{1-\sigma} / (1 - \sigma)
$$

for $\sigma \neq 1$. Use the above Euler equation to show that the optimal savings function is

$$
s_t = s(w_t, r_{t+1}) = \frac{(1 + r_{t+1})^{(1-\sigma)/\sigma}}{(1 + \theta)^{1/\sigma} + (1 + r_{t+1})^{(1-\sigma)/\sigma} w_t}
$$

This expression shows that there are conflicting effects of the interest rate on savings; the income and substitution effects. Raising the interest rate increases the value of the first period income, which increases consumption today (income effect). This reduces savings. But a higher interest rate also raises the rate of return to savings or equivalently, raises the price of consuming today, causing consumption to fall (substitution effect). With log preferences these effects cancel out.

Define the inter-temporal elasticity of substitution ITES $= 1/\sigma$. Show that if the ITES is large then the substitution effect dominates, so that $s$ tends to rise in response to a rise in $r$.

**The Firm’s Problem** The firm takes prices as given and chooses factor inputs to solve:

$$
\max_{K_t, N_t} \{ F(K_t, N_t) - r_t K_t - w_t N_t \}
$$

The FOC are:

$$
F_K = r_t
$$

$$
F_N = w_t
$$

In per capita terms this translates to:

$$
\dot{f}(k_t) = r_t
$$

and, using the fact that CRS implies zero profits:

$$
w_t = f(k_t) - k f'(k_t)
$$
Equilibrium A Recursive Competitive Equilibrium of this economy prices \( \{w_t, r_t\}_{t=0}^{\infty} \), a savings policy function \( s(w, r) \), and factor demands \( n = g_n(w, r) \) and \( k = g_k(w, r) \) such that

1. (Optimality conditions):
   a. the savings policy \( s(w, r) \) maximizes \( u(c_t, c_{t+1}) \) for each generation \( G_t \), given \( r = r_{t+1} \), and subject to the household budget constraint.
   b. for \( t = 0 \), \( c_0^0 = e_2 \); the initial old in \( G_0 \) consume their endowments
   c. the factor demand functions \( n = g_n(w, r) \) and \( k = g_k(w, r) \) solve the firm’s maximization problem

2. Markets clear:
   a. the FC constraint is satisfied with equality in each period.
   b. the supply of savings equals the demand:
      \[
      (1 + n) k_{t+1} = s(w_t, r_{t+1})
      \]

In general, one can use the factor-price conditions to write capital next period as a function of capital in the current period; this defines a law of motion for capital:

\[
(1 + n) k_{t+1} = s(f(k_t) - kf'(k_t), f'(k_t))
\]

5.3 Steady-State
Here we deal only with the simplified case with:

1. log utility: \( u(c) = \ln c \)
2. CD production: \( f(k) = Ak^\alpha \)

These assumptions imply a simple for savings as income and substitution effects cancel out:

\[
s(w, r) = \frac{w}{2 + \theta}
\]

and hence

\[
k^*_t = \frac{s(w_t, r_t) \frac{1}{1+n} (1 - \alpha) Ak_t^\alpha}{1+n}
\]

At the steady-state we have

\[
k^* = \left( \frac{\frac{1}{2 + \theta} (1 - \alpha) A}{1+n} \right)^\frac{1}{1-\alpha}
\]
5.3.1 Dynamic Efficiency

This in turn implies that \( k^\ast \) can be higher \( k_{GR} \) and hence the economy can be dynamically inefficient:

\[
 f'(k^\ast) = \alpha A k^{\ast \alpha - 1} = \alpha (1 + n) \frac{2 + \theta}{(1 - \alpha)} \geq \delta + n
\]

In this situation, a reduction in the savings rate increases aggregate consumption in all generations \( G_t \) where \( t > 0 \). But the feasibility constraint implies that the initial old will lose consumption and hence be worse off. This cannot be an equilibrium because it requires the initial old to consume less. However since aggregate consumption increased, it would be possible for the initial young to compensate the initial old for this and still be better off. Hence with transfers one can attain a better allocation.

Another way to put this is that the young save too much because there is no other mechanism to provide retirement consumption unless the govt implements a transfer program.

5.4 Fiscal Policy

Consider now the effect of balanced-budget government spending on output. Suppose that the spending \( g_t \) is financed by lump-sum taxes \( \tau_t \) on the young. The income of the young falls to \( w_t - \tau \). With log utility and CD production this implies

\[
 s_t = \frac{w_t - \tau}{2 + \theta}
\]

and the Steady-state capital is

\[
 k^\ast = \left( \frac{1 - \alpha}{2 + \theta} \left( 1 - \alpha \right) A - g \right)^{\frac{1}{1 - \alpha}}
\]

So the long-run effect of government spending is to reduce output, because it causes people to save less.

Exercise: Show that if instead the government taxed the old to pay for \( g_t \), then output would increase.

Now suppose that the government issued debt \( b_t \) to reduce taxes. The GBC is

\[
 (1 + r_t) b_t + g = \tau + b_{t+1} (1 + n)
\]

The asset-market clearing condition is now

\[
 s_t = (1 + n) (k_{t+1} + b_{t+1})
\]

This says that total savings now has to cover govt borrowing as well as capital.

For the simple log-utility example we have been considering, we can show that the law of motion for capital is, for the case where \( b_t = b \) is constant

\[
 k_{t+1} = \frac{1 + \alpha}{2 + \theta} \left( 1 - \alpha \right) A k^\alpha_t - \tau}{1 + n} - b
\]
This implies that financing government spending with bonds actually reduces capital and causes output to fall in the long run. The reduction in taxes causes savings to rise, but as the debt rises by more than savings, capital accumulation declines.

With $\theta \neq 1$ there would be an additional effect that of the change in interest rate on savings. The big picture is that there are real effects here of how the government pays for spending, because unlike the IH case, deferring taxes reduces the present value of taxes for people who only live two periods.

5.5 Social Security

In a public pension system the govt awards the current old people a benefit $P$. There are two ways to pay for this. Both require taxing the young, say via a lump-sum tax $\tau$. In the first, the fully-funded method, the tax revenues collected at time $t$ from the young Generation $G_t$ are invested by the government and the proceeds used at time $t+1$ to pay the old of generation $G_t$. A fully-funded public-pension system essentially replaces private savings with government savings and so has no effect provided that the tax rate is not higher than the optimal savings rate.

The interesting case is the "pay as you go" PAYG system. In the PAYG method, the tax revenues collected at time $t$ from the young generation $G_t$ are immediately transferred to the old of generation $G_{t-1}$, who each receive $(1 + n)\tau$.

The second-period budget constraint becomes

$$c_{t+1} = (1 + r_{t+1}) s_t + (1 + n)\tau$$

Consider the log example from the previous section. Its easy to show that

$$s_t = \frac{1}{2 + \theta} \left[ w_t - \frac{1 + n}{1 + r} (1 + \theta) \tau \right]$$

so that, using the market-clearing conditions

$$k_{t+1} = \frac{s_t}{1 + n} = \frac{1}{(1 + n)(2 + \theta)} \left[ (1 - \alpha) k_t^\alpha - \tau \left( 1 + \frac{(1 + \theta)(1 + n)}{\alpha k_{t+1}^{\alpha-1}} \right) \right]$$

while this is a little tricky to solve for the steady-state $k^*$ it is clear that the tax reduces capital next period, and hence shifts down the graph of $k_{t+1}(k_t)$ which implies a lower steady state.

Recalling the Golden Rule results, this implies that if, under the initial steady state, $r > n$ then the public pension policy makes future generations worse off by reducing steady-state consumption. On the other hand everyone may be able to gain from this policy if the economy is initially saving at such a high rate that $r < n$. Long-run consumption is increased because some of what would be saved in the absence of the policy is now consumed by the old; the reduction in ss output caused by PAYG is less than the reduction in savings.
**Dynamic Efficiency**  Recall the result for the golden-rule capital stock: $f'(k_{GR}) = n$. Applying the equilibrium condition on $r$ (ie old agents consume *all* of the capital stock), we get $r = n$ for the condition that the steady-state capital stock implied by the golden-rule savings rate must solve. Economics undergrads learn that saving more than this rate is inefficient because it reduces consumption in both the short and the long runs. The inefficiency here is *dynamic* because it involves comparison over time rather than across agents.

The Ramsey model showed us that even savings rates lower than the golden rule can be dynamically inefficient if the household has a rate of time preference greater than one.

In the OLG model, the dynamic efficiency results from the existence of a redistribution technology that dominates the savings technology for the transfer of resources over time if savings is too high.